Differential Geometry and its Applications 5 (1995) 205–218
North-Holland

Reconstruction of domains from their groups of quasiconformal autohomeomorphisms

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Communicated by S. Gindikin
Received 25 July 1993

Abstract: For an open set $U \subseteq \mathbb{R}^n$, let $QC(U)$ denote the group of all quasiconformal homeomorphism of U. The following is our first main result. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, and suppose that τ is a group isomorphism between $QC(U)$ and $QC(V)$. Then there is a quasiconformal homeomorphism φ from U onto V such that φ induce τ. That is, for every $f \in QC(U)$: $\tau(f) = \varphi \circ f \circ \varphi^{-1}$.

Keywords: Homeomorphism groups, reconstruction, Lipschitz, Sobolev spaces.
MS classification: 30C60, 46B99, 54H99.

Introduction

A homeomorphism φ between two open sets $U, V \subset \mathbb{R}^n$, $n > 1$ ($\varphi : U \cong V$) is called a *quasiconformal* (QC) homeomorphism, if there is $K \geqslant 1$ such that for every $x \in U$ there is $r > 0$ such that for every $y_1, y_2 \in U$

$$\text{if} \quad |y_1 - x| = |y_2 - x| < r \quad \text{then} \quad \frac{|\varphi(y_2) - \varphi(x)|}{|\varphi(y_1) - \varphi(x)|} \leqslant K.$$

Let $QC(U) \stackrel{\text{def}}{=} \{f : U \cong U \mid f \text{ is } QC\}$. By [6], $QC(U)$ is a group. That is, if $f, g \in QC(U)$, then f^{-1}, $f \circ g \in QC(U)$.

We shall not deal with the case of $\mathbb{R}^n$ and $\mathbb{R}^m$ $n = 1$. So we assume that whenever $\mathbb{R}^n$ and $\mathbb{R}^m$ are mentioned, $n, m > 1$. In Section 1 we prove the following theorem.

Theorem 0.1. *For $m, n > 1$ let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, and suppose that τ is a group isomorphism between the groups $QC(U)$ and $QC(V)$. Then there is $\varphi : U \cong V$ such that φ is QC and φ induces τ. That is, for every $f \in QC(U)$:*

$$\tau(f) = \varphi \circ f \circ \varphi^{-1}.$$

The proof of the theorem has two main steps. The first step is the claim that if $\tau : QC(U) \cong QC(V)$ is a group isomorphism, then there is $\varphi : U \cong V$ which induces τ. This claim is a special case of the following theorem appearing in [3, Theorem 3.5(c)].

Theorem 0.2. ([3, Theorem 5(c)]) For $i = 1,2$, let X_i be a locally compact Hausdorff space, and G_i be a group of homeomorphisms of X_i such that for every $x \in X_i$ and open $U \ni x$ the set

$$A \stackrel{\text{def}}{=} \{g(x) \mid g \in G_i \text{ and } g\lceil(X_i - U) = \mathrm{Id}\}$$

is somewhere dense. That is, $\mathrm{int}(\mathrm{cl}(A)) \neq \emptyset$. Let $\tau : G_1 \cong G_2$ be a group isomorphism. Then there is $\varphi : X_1 \cong X_2$ such that φ induces τ.

For an open subset $U \subseteq \mathbb{R}^n$, $n > 1$, let

$$LIP(U) \stackrel{\text{def}}{=} \{f : U \cong U \lceil f \text{ and } f^{-1} \text{ are Lipschitz}\}.$$

The second step in the proof of Theorem 0.1 is

Theorem 0.3. *Let $U, V \subseteq \mathbb{R}^n$, $n > 1$ be open and $\varphi : U \cong V$. Suppose that for every $f \in LIP(U)$, $\varphi \circ f \circ \varphi^{-1} \in QC(V)$. Then φ is quasiconformal.*

Indeed, in order to prove Theorem 0.1 we only need the following weaker statement:

$$\text{if for every } f \in QC(U)\ \varphi \circ f \circ \varphi^{-1} \in QC(V) \text{ then } \varphi \text{ is } QC.$$

However, the statement of Theorem 0.3 is useful in proving e.g., that $QC(U)$ can't be isomorphic to $LIP(V)$.

Similar questions can be asked, of course, for many other types of homeomorphism groups. In this work we prove also the local version of Theorem 0.1. That is, if $LQC(U)$ denotes the group of locally quasiconformal homeomorphisms of U then an isomorphism between $LQC(U)$ and $LQC(V)$ is always induced by a locally quasiconformal homeomorphism between U and V (i.e., the class LQC is faithful). It is the local counterpart of Theorem 0.3 that seems to be interesting in this context. The same is correct for manifolds.

The first result on reconstruction of topological spaces from their homeomorphism groups was proved by J. Whittaker [7]. He proved that the class of Euclidean manifolds is faithful. More precisely, the class $\{(X, H(X)) \mid X \text{ is a Euclidean manifold}\}$ is faithful.

Rubin [3] proved the faithfulness of various other classes of topological spaces like Euclidean manifolds with boundary, polyhedra and manifolds over normed vector spaces.

W. Ling [2] proved that

(1) For every $\leqslant r \leqslant \infty$ a group isomorphism between $\mathrm{Diff}^r(X)$ and $\mathrm{Diff}^r(Y)$ (the groups of C^r-homeomorphism of X and Y, respectively) is induced by a C^r-homeomorphism between X and Y.

In addition he proved that

(2) For various structures on manifolds like foliations, symplectic forms, volume forms and others, an isomorphism between the groups of diffeomorphism preserving those structures is induced by a diffeomorphism of the same type.

The special case of Ling's result (1) for $\mathrm{Diff}^\infty(X)$ was proved earlier by Takens in [5].

Yomdin and Rubin [4] strengthened Ling's result (1) in several directions. In particular, they deal with Lipschitz homeomorphisms groups, and extended some of the results to Banach spaces.

Unfortunately, Ling's works seem not to have been published until now. Filipkiewicz [1] proved the reconstruction results for groups of the form $\mathrm{Diff}^r(X)$. These results were obtained earlier by Ling and independently by the authors. Filipkiewicz was apparently unaware of Ling's works, but quoted the results of [4].

1. Preliminaries

1.1. *Bilipschitz homeomorphisms and quasiconformal homeomorphisms*

Let $\varphi : U \cong V$ be a homeomorphism between open sets U, $V \subset \mathbb{R}^n$, let a ball $B(x,r) \subset U$. We will use following quantities:

$$L_\varphi(x,r) = \max_{|y-x|=r} |\varphi(y) - \varphi(x)|,$$

$$l_\varphi(x,r) = \min_{|y-x|=r} |\varphi(y) - \varphi(x)|,$$

the metrical dilatation of φ in $B(x,r)$:

$$K_\varphi(x,r) = \frac{L_\varphi(x,r)}{l_\varphi(x,r)},$$

the metrical dilatation of φ at x:

$$K_\varphi = \lim_{r\to 0} \sup K_\varphi(x,r),$$

the metrical dilatation of φ:

$$K(\varphi) = \sup_{x\in G} K_\varphi(x).$$

A homeomorphism φ is quasiconformal iff $K(\varphi) < \infty$.

We will use also the following quantities:

$$\overline{KI}(\varphi) = \sup_{x,y\in G} \frac{|\varphi(x) - \varphi(y)|}{|x-y|},$$

$$\underline{KI}(\varphi) = \inf_{x,y\in G} \frac{|\varphi(x) - \varphi(y)|}{|x-y|},$$

$$KI(\varphi) = \max(\overline{KI}(\varphi), \underline{KI}^{-1}(\varphi)).$$

If $\overline{KI}(\varphi) < \infty$ then the homeomorphism φ is called a *Lipschitz homeomorphism.* If $\underline{KI}(\varphi) > 0$ then φ^{-1} is a Lipschitz homeomorphism. If $KI(\varphi) < \infty$ then φ is a *bilipschitz homeomorphism.* We denote by $LIP(U,V)$ the set of all bilipschitz homeomorphisms between U and V and by $LIP(U) \stackrel{\mathrm{def}}{=} LIP(U,U)$. $LIP(U)$ is regarded as a group.

Claim 1.1. (i) *If* $\varphi \in LIP(\mathbb{R}^n)$ *and* τ *is an isometry, then* $\tau \circ \varphi \in LIP(\mathbb{R}^n)$ *and* $KI(\tau \circ \varphi) = KI(\varphi)$.

(ii) *If* $s : \mathbb{R}^n \to \mathbb{R}^u$ *is a similarity, (i.e., if* $s(x) = Kx$, $K \neq 0$*) then for each* $\varphi \in LIP(\mathbb{R}^n)$, $s \circ \varphi \circ s^{-1} \in LIP(\mathbb{R}^n)$ *and* $KI(s \circ \varphi \circ s^{-1}) = KI(\varphi)$.

Proof. (i) is evident.

(ii) If s is a similarity $s(x) = Kx$, then

$$\begin{aligned}|s \circ \varphi \circ s^{-1}(x) - s \circ \varphi \circ s^{-1}(y)| &= \left|K\varphi\Big(\frac{1}{K}x\Big) - K\varphi\Big(\frac{1}{K}y\Big)\right| \\ &= K\left|\varphi\Big(\frac{1}{K}x\Big) - \varphi\Big(\frac{1}{K}y\Big)\right| \leqslant K \cdot KI(\varphi)\left|\frac{1}{K}x - \frac{1}{K}y\right| \leqslant KI(\varphi)|x-y|.\end{aligned}$$

So $KI(s\circ\varphi\circ s^{-1}) \leqslant KI(\varphi)$. From the other side if $\varphi_1 = s\circ\varphi\circ s^{-1}$ then $\varphi = s^{-1}\circ\varphi_1\circ s$ and $KI(\varphi) \leqslant KI(\varphi_1) = KI(s\circ\varphi\circ s^{-1})$. Hence

$$KI(\varphi) = KI(s \circ \varphi \circ s^{-1}). \quad \square$$

1.2. *Quasitranslation*

We will construct a bilipschitz homeomorphism, that coincides with translation on some ball and coincides with identity outside some bigger concentric ball.

At first we consider the two-dimensional case.

Let $x_1 = (-\frac{1}{2}, 0)$ and $x_2 = (\frac{1}{2}, 0)$ be two points on the plane $\mathbb{R}^2$; $B_0 = B(0, \sqrt{2})$, $B_1 = B(x_1, \frac{1}{4})$ and $B_2 = B(x_2, \frac{1}{4})$. We will construct a diffeomorphism $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(\varphi\restriction B_1)\,(x,y) = (x+1, y)$, $\varphi(B_1) = B_2$, $\varphi\restriction(\mathbb{R}^2\backslash B_0) = \mathrm{Id}$, (i.e., $\varphi\restriction B$ is the translation $\tau_1(x,y) = (x+1,y)$). Of course, φ_2 is a bilipschitz homeomorphism.

With the help of a decomposition of unit it is possible to construct two functions ψ and θ such that:

1) $\psi : \mathbb{R} \cong \mathbb{R}$; $\psi \in C^\infty(\mathbb{R})$; $\psi(x) = x$ for $x \in (-\infty, -1) \cup (1, \infty)$; $\psi(x) = x + 1$ for $x \in (-\frac{3}{4}, -\frac{1}{4})$, for all other x's ψ is monotone function with positive derivative;

2) $\theta : \mathbb{R} \cong \mathbb{R}$; $0 \leqslant \theta(y) \leqslant 1$; $\theta(y) = 1$ for $|y| > \frac{1}{2}$; $\theta(y) = \theta(-y)$; $\theta(y) = 0$ for $|y| < \frac{1}{4}$; $\theta(y)$ is a C^∞-function.

Then $\varphi_2(x,y) = ((1-\theta(y))\psi(x) + \theta(y)x, y)$. The mapping φ is a diffeomorphism, because the *Jacobian*

$$J(x,y) = (1-\theta(y))\psi'(x) + \theta(y) > 0.$$

It is evident that $|\varphi_2(x,y) - (x,y)| \leqslant 2\sqrt{2}$. The diffeomorphism φ_2 has the prescribed properties:

$$\begin{aligned}&\text{for } (x,y) \in (-\tfrac{3}{4}, -\tfrac{1}{4}) \times (-\tfrac{1}{4}, \tfrac{1}{4}), \quad \varphi(x,y) = (x+1, y),\\ &\text{and for } |x| \geqslant 1,\ |y| \geqslant 1, \quad \varphi(x,y) = (x,y).\end{aligned}$$

With the help of φ_2 we will construct a corresponding n-dimensional homeomorphism.

Let $\bar{x}^1 = (-\frac{1}{2}, 0, \ldots, 0)$, $\bar{\bar{x}} = (\frac{1}{2}, 0, \ldots, 0)$, $B_0 = B(0, \sqrt{2})$, $B_1 = B(\bar{x}, \frac{1}{4})$, $B_2 = B(\bar{\bar{x}}, \frac{1}{4})$, $\varphi_n(x) = \varphi_2(x_1, \sqrt{x_2^2 + \ldots + x_{n-2}^2})$. Then the diffeomorphism φ_n has following properties:

$$(\varphi_n \restriction B_1)(x) = \tau_1(x) = (x_1 + 1, x_2, \ldots, x_n), \quad \varphi_n \restriction (\mathbb{R}^n \setminus B_0) = \mathrm{Id}\,.$$

We will use notation $Q(n) \stackrel{\text{def}}{=} KI(\varphi_n)$, $\rho(x, A)$ for a distance between point x and set A.

Lemma 1.2. *Let U be a domain in $\mathbb{R}^n$; let x', x'' be two points in U such that*

$$|x' - x''| < \tfrac{1}{2}\rho(\tfrac{1}{2}(x' + x''), \partial U)$$

Then there exists a diffeomorphism $\varphi : U \to U$ such that

$$(\varphi \restriction B(x', \tfrac{1}{4}|x' - x''|))(x) = x + (x'' - x');$$

$$\varphi \restriction \Big(U \setminus B\Big(\tfrac{1}{2}|x' + x''|, \frac{1}{\sqrt{2}}\rho\big(\tfrac{1}{2}|x' + x''|, \partial U\big)\Big)\Big) = \mathrm{Id} \quad \textit{and} \quad KI(\varphi) = Q(n).$$

Proof. Denote $\rho_0 = \rho(\frac{1}{2}|x' + x''|, \partial U)$, $\rho_1 = \frac{1}{4}|x' + x''|$. The ball $B(\frac{1}{2}(x' + x''), \rho_0) \subset U$. We consider three mappings: a translation $\tau(x) = x - \frac{1}{2}(x' + x'')$; a rotation $\theta(x)$, such that $\theta(x'') = (|x''|, 0, \ldots, 0)$ and the similarity $S(x) = x/|x' - x''|$. It is evident that $S(B(0, \rho_0)) \supset B(0, 2)$. Consider the diffeomorphism:

$$\varphi = (s \circ \theta \circ \tau)^{-1} \circ \varphi_n \circ (s \circ \theta \circ \tau) = (\tau^{-1} \circ \theta^{-1}) \circ (s^{-1} \circ \varphi_n \circ s) \circ (\theta \circ \tau).$$

We will use the notation $\tilde{x}' = (-\frac{1}{2}|x' - x''|, 0, \ldots, 0)$, $\tilde{x}'' = (\frac{1}{2}|x' - x''|, 0, \ldots, 0)$, $|x' - x''| = \rho_1$. Then $(\theta \circ \tau)(x') = \tilde{x}'$, $(\theta \circ \tau)(x'') = \tilde{x}''$, $(\theta \circ \tau)(B(x', \rho_1)) = B(\tilde{x}', \rho_1)$, $(\theta \circ \tau)(B(x'', \rho_1)) = B(\tilde{x}'', \rho_1)$, $S(\tilde{x}') = (-\frac{1}{2}, 0, \ldots, 0)$, $S(\tilde{x}'') = (\frac{1}{2}, 0, \ldots, 0)$,

$$s^{-1} \circ \tau_1 \circ s = |x' + x''|\Big(\frac{x_1}{|x' - x''|} + 1, \frac{x_2}{|x' - x''|}, \ldots, \frac{x_n}{|x' - x''|}\Big).$$

So $s^{-1} \circ \varphi_n \circ s$ is the translation on the ball $B(\tilde{x}', \rho_1)$ and $s^{-1} \circ \varphi_n \circ s$ is identity on the set $\mathbb{R}^n \setminus B(\frac{1}{2}|x' + x''|, \sqrt{2}\rho_1)$. From Claim 1.1 it follows that

$$KI(\varphi) = KI(\varphi_n). \quad \square$$

1.3. *Two types of quasi-rotations*

We will construct a bilipschitz homeomorphism, that coincides with rotations inside a ball and coincides with identity outside a bigger concentric ball. Except that we need a bilipschitz homeomorphism that coincides with identity inside of a ball, outside of a bigger concentric ball and coincides with a rotations inside a concentric spherical layer that belongs to the big ball.

We will use the notation B_k for $B(0, k)$, $SO_n(x)$ for the group of all orientation preserving orthogonal transformations with fixed point x.

Lemma 1.3. *Let U be a domain in $\mathbb{R}^n$, $x \in G$, $\rho < \rho(x, \partial U)$, $\psi \in SO_n(x)$. Then there exist a diffeomorphism $\varphi : U \to U$ such that:*

$$\varphi\lceil(B(x, \tfrac{1}{2}\rho_0)) \equiv \psi, \quad \varphi\lceil U \backslash B(x, \rho_0) \equiv \mathrm{Id} \quad \text{and} \qquad KI(\varphi) = Q(n),$$

where a number $Q(n)$ depends only on n. □

Proof is similar to the proof of Lemma 1.2.

Lemma 1.4. *Let U be a domain in $\mathbb{R}^n$, $x \in U$, $\rho_0 < \rho(x, \partial U)$, $\psi \in SO_n(x)$. Then there exists a diffeomorphism $\varphi : U \to U$ such that $\varphi\lceil(B(x_0, \frac{1}{4}\rho_0)) \equiv \mathrm{Id}$, $\varphi\lceil(U \backslash B(x_0, \rho_0)) \equiv \mathrm{Id}$, $\varphi\lceil(B(x, \frac{3}{4}\rho_0) \backslash B(x, \frac{1}{2}\rho_0)) \equiv \psi$ and $KI(\varphi) = \tilde{Q}_n$, where a number $\tilde{Q}_n$ depends only on n.*

Proof is similar to the proof of Lemma 1.2.

2. Local version of the main theorem

In this section we will prove local version of the main theorem (Theorem 2.5) for the group LQC (locally quasiconformal homeomorphisms).

A homeomorphism $\varphi : G \simeq G$ between two domains in $\mathbb{R}^n$ is called locally quasiconformal (LQC) if for every $x \in G$ there is a neighbourhood $U(x)$ of x such that $\varphi\lceil U(x)$ is quasiconformal. Of course the set $LQC(G)$ of all locally quasiconformal homeomorphisms $\varphi : G \simeq G$ is a group under composition. Obviously, the following inclusion holds:

$$LQC(G) \supset QC(G) \supset LIP(G).$$

2.1. *Preliminary result*

Theorem 2.1. ([6]) *Let $\varphi : G \simeq G'$ be a quasiconformal homeomorphism, G, $G' \subset R^n$. Suppose that a closed ball $\overline{B(x, 2r)} \subset G$ and $B(\varphi(x), L_\varphi(x, 2r)) \subset G'$. Then*

$$K_\varphi(x, r) \leqslant \exp(Q_1(n) \cdot K(\varphi)),$$

where $Q_1(n)$ depends only on n.

2.2. *Movable groups of homeomorphisms*

Let X be a locally compact Hausdorff space and g be a group of homeomorphisms of X. The group g is called *movable* [3] if for every $x_0 \in X$ and every neighbourhood $U(x_0)$ of x_0 the set

$$A_0 \stackrel{\text{def}}{=} \{g(x_0) | g \in G \text{ and } g\lceil(x \backslash U(x_0)) = \mathrm{Id}\}$$

is somewhere dense. Remember that a set A is somewhere dense if $\mathrm{Int}(\overline{A}) \neq \emptyset$.

Lemma 2.2. *Let G be a domain in $\mathbb{R}^n$. The groups $LQC(G)$, $QC(G)$, $LIP(G)$ are movable.*

Proof. $LIP(G)$ is the smallest group. If $LIP(G)$ is movable, then $QC(G) \supset LIP(G)$ and $LQC(G) \supset LIP(G)$ are also movable. Let $x_0 \in G$ and $U(x_0)$ is an arbitrary neighbourhood of x_0. Choose a ball $B(x_0, r)$ such that $\overline{B}(x_0, 2r) \subset U(x_0)$. Hence, by Lemma 1.2 for each point $z \in (x_a, \frac{1}{4}r)$ there exists a homeomorphism $g_z \in LIP(G)$ such that $g_z(x_0) = z$ and $g_z(x) \equiv x$ for all $x \in G \backslash B(x, 2r)$. So $A_0 \supset B(x, \frac{1}{4}r)$ and it is somewhere dense. □

From Lemma 2.2 and Theorem 0.2 it follows

Theorem 2.3. *Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open and $\tau : QC(U) \cong QC(V)$ be a group isomorphism, then there is $\varphi : U \cong V$ such that φ induces τ.*

2.3. *The main observation*

We will use the standard notation $S(x, r)$ for a sphere with a center x and a radius r.

Lemma 2.4. *Let $V \subset \mathbb{R}^n$ be open, $S \subset V$ be closed, $x \in V$, $y_0, y_1 \in S$,*

$$r_0 = \min(|y - x|;\ y \in S) = |y_0 - x|,$$

$$r_1 = \max(|y - x|;\ y \in S) = |y_1 - x|.$$

Let $K_1 = r_1/r_0$. Suppose that $r_2 > r_1$ and $\overline{B(x, r_2)} \subset V$. Let $f : V \cong V$ and $K > 0$ be such that for every $0 < r \leqslant r_2$, $K_\varphi(x, r) \leqslant K$. Suppose also that $f(x) = x$, $f(y_1) = y_0$, $f(S) = S$ and that there is $z^* \in B(x, r_2) \backslash \overline{B(x, r_1)} \overset{\text{def}}{=} D_0$ such that $f(z^*) \subset D_0$. Then $K \geqslant K_1$.

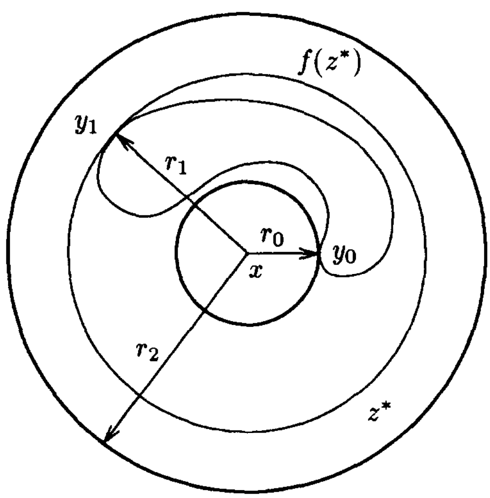

Fig 1.

Proof. By contradiction, let $K_1 > K$. Hence $r \overset{\text{def}}{=} K \cdot r_0 < r_1$. We will prove that in this case $f(\overline{B(x,r_1)}) \subseteq \overline{B(x,r)}$.

For every $y \in \partial B(x, r_1)$

$$|f(y) - x| = |f(y) - f(x)| \leqslant K|f(y_1) - f(x)| \leqslant K|y_0 - x| = Kr_0 < r.$$

So $f(S(x,r_1)) \subseteq \overline{B(x,r)}$.

Suppose by contradiction that $y \in B(x,r_1)$ and $|f(y) - x| > r$. Let

$$z = x + r_1 \frac{y-x}{|y-x|}.$$

Then $z \in S(x,r_1)$. Let $a > 0$ be such that $r + a < r_2$. Hence there is $t \in (0,1]$ such that

$$r < |f(ty + (1-t)z) - x| < r + a.$$

Let $y' = ty + (1-t)z$. From our construction

$$|y' - x| < r_1 < |z^* - x|$$

and

$$r < |f(y') - x|, \quad r_2 > |f(z^*) - x|.$$

Let I be a path that connects $f(y')$ and $f(z^*)$ in the spherical layer $B(x,r_2)\backslash\overline{B(x,r)}$. From one side $I \cap f(S(x,r_1)) = \emptyset$, because $f(S(x,r_1)) \subseteq \overline{B(x,r)}$. From the other side $|y' - x| < r_1 < |z^* - x|$ and $f^{-1}(I) \cap S(x,r_1) \neq \emptyset$. So we proved that $f(\overline{B(x,r_1)}) \subseteq \overline{B(x,r)}$.

Remember that $S \subset \overline{B(x,r_1)}$. So $f(S) \subset \overline{B(x,r)}$. But $y_1 \in S\backslash\overline{B(x,r)}$. This contradicts the fact that $f(S) = S$. $\square$

2.4. *The main theorem for homeomorphisms (local version)*

Theorem 2.5. *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$ be open and $\varphi : U \cong V$ be a homeomorphism. Suppose that for each autohomeomorphism $f \in LIP(U)$ the autohomeomorphism*

$$f^{\varphi} = \varphi \circ f \circ \varphi^{-1}$$

belongs to $LQC(U)$. Then $\varphi \in LQC(U,V)$.

Proof. By contradiction. Suppose that φ is not locally quasiconformal. Hence there is a point $x \in U$ and a ball $B = B(x,r)$ such that $\overline{B} \subset U$ and $K(\varphi\restriction B) = \infty$. So

$$\sup_{x \in B} K_{\varphi}(x) = \infty.$$

Therefore there is a sequence $\{x_k \subset B\}$ such that $K_{\varphi}(x_k) \to \infty$ if $k \to \infty$. We can suppose that

$$\lim_{k \to \infty} x_k = x_0 \quad \text{and} \quad x_0 \subset B.$$

There are two possible cases:

1) There is a subsequence $\{x_{k_i}\}$ of the sequence $\{x_k\}$ such that $x_{k_i} \neq x_{k_j}$ for all $i \neq j$.

2) There is a number k_0 such that $x_k = x_0$ for all $k > k_0$.

Proof for case 1. Without loss of generality we can suppose that $x_{k_1} \neq x_{k_2}$ for all $k_1 \neq k_2$, $K_k \stackrel{\text{def}}{=} K_\varphi(x_k) \to \infty$ for $k \to \infty$. Choose a sequence of balls $\{K_k = B(x_k, r_k)\}$ with the following properties:

a) $B_k \cap B_{k_1} = \emptyset$ if $k \neq k_1$;

b) $U_k \stackrel{\text{def}}{=} B(\varphi(x_k), 2L_\varphi(x_k, r_k)) \subset V$.

Inside each ball B_k we choose a new ball $\hat{B}_k = B(x_k, \rho_k) \subset B(x_k, \frac{1}{2}r_k)$ such that:

$$Q_k = K_\varphi(x_k, \rho_k) > \tfrac{1}{2}K_k.$$

So $\lim\limits_{k\to\infty} Q_k = \infty$. Remember that $Q_k = L_k/l_k$, where

$$L_k = L_\varphi(x_k, \rho_k),$$
$$l_k = l_\varphi(x_k, \rho_k).$$

For each k, choose two points u_k and z_k such that $u_k, z_k \subset \partial\hat{B}_k$ and

$$|\varphi(u_k) - \varphi(x_k)| = L_k,$$
$$|\varphi(z_k) - \varphi(x_k)| = l_k.$$

Let $\theta \in SO_n(x_k)$ and $\theta_k(z_k) = u_k$. By Lemma 1.3 there is a diffeomorphism $f_k : U \cong U$, such that $f_k\lceil\hat{B}_k = \theta_k$, $f_k\lceil(U\backslash B_k) = \mathrm{Id}$, $KI(f_k) \leqslant Q(n)$, where the number $Q(n)$ depend only on n. The diffeomorphism $f_0 : U\backslash\{x_0\} \cong U\backslash\{x_0\}$ such that $f_0(x) = f_k(x)$ for $x \in B_k$ and all k and $f_0(x) = x$ for all other x is also bilipschitz. It is evident that $KI(f_0) \leqslant Q(n)$. A point x_0 is a removable singularity for bilipschitz homeomorphism. So we can extend f_0 to the point x_0 without any change of dilatation. Hence $g_0 = f_0^\varphi$ is quasiconformal, i.e., $Q_0 = K(g_0) < \infty$. Remember that $f_0\lceil\hat{B}_k = f_k$. Hence $g_k = f_k^\varphi$ is quasiconformal and $Q_k = K(g_k) \leqslant Q_0$. Let $V_k = \varphi(B_k)$, $S_k = \varphi(\hat{B}_k)$. Then g_k has the following properties:

a) $g_k(V_k) = V_k$,

b) $g_k(S_k) = S_k$,

c) $g_k\lceil(V\backslash\varphi(B_k)) = \mathrm{Id}$,

d) $g_k(\varphi(z_k)) = \varphi(u_k)$,

e) $g_k(\varphi(x_k)) = \varphi(x_k)$.

Remember, that $V_k \subset B(\varphi(x_k),\ L_\varphi(x_k, \rho_k)) \subset B(\varphi(x_k), 2L(x_k, \rho_k)) \stackrel{\text{def}}{=} U_k \subset V$. So all conditions of Theorem 2.1 hold for g_k. Therefore, for all $\rho \geqslant \rho_k$ the following uniform estimate is true:

$$K_{g_k}(\varphi(x_k), \rho) = \frac{L_{g_k}(\varphi(x_k), \rho)}{l_{g_k}(\varphi(x_k), \rho)} < P < \infty, \tag{2.2}$$

where the constant P depends only on n and Q_0.

For the sets V_k, S_k, points $\varphi(x_k)$, $\varphi(u_k)$, $\varphi(z_k)$, homeomorphism g_k, numbers $r_0 = l_k$, $r_1 = L_k$, $K_1 = Q_k$, $r_2 = 2L_k$ and an arbitrary point

$$z^* \subset U_k \backslash \overline{B(\varphi(x_k), L_\varphi(x_k, \rho_k))},$$

all conditions of the Lemma 2.4 are true. By this lemma $K_{g_k}(\varphi(x_k), \rho_k) \geqslant Q_k$. So $K_{g_k}(\varphi(x_k), \rho) \to \infty$. This conclusion contradicts to inequality (2.2). So φ is locally quasiconformal.

Proof for case 2. We give only a sketch of the proof, because the difference between proofs for case 1 and case 2 concerns only a construction of a quasiconformal homeomorphism that is similar to f_0 in the proof for case 1.

In case 2 the homeomorphism φ is not quasiconformal at the point x_0, i.e. $K_\varphi(x_0) = \infty$. Therefore there is a sequence of balls $\{B_k = B(x_0, r_k)\}$ such that:

a) $r_k \to 0$ if $k \to \infty$,

b) $8r_{k+1} < r_k$,

c) $K_k = K_\varphi(x, r_k) \to \infty$ if $k \to \infty$,

d) $\overline{B(x_0, 2r_1)} \subset U$,

e) $V_0 = B(\varphi(x_0), 2L_\varphi(x_0, r_1)) \subset V$.

For each k choose two points u_k and z_k such that u_k, $z_k \subset \partial B_k$ and

$$|\varphi(u_k) - \varphi(x_0)| = L_\varphi(x_0, r_k);$$
$$|\varphi(z_k) - \varphi(x_0)| = l_\varphi(x_0, r_k).$$

Let $\theta \in SO_n(x_0)$ and $\theta_k(z_k) = u_k$. By Lemma 1.4 there is a diffeomorphism f_k : $U \cong U$ such that:

a) $f_k \restriction U \backslash \overline{B(x_0, 2r_k)} = \mathrm{Id}$,

b) $f_k \restriction B \backslash B(x_0, \frac{r_k}{2}) = \mathrm{Id}$,

c) $f_k \restriction B\left(x_0, \frac{3}{2}r_k\right) \backslash B(x_0, r_k) = \theta_k$,

d) $KI(f_k) < \tilde{Q}(n)$,

where the number $\tilde{Q}(n)$ depends only on n. The diffeomorphism $f_0 : U \backslash x_0 \cong U \backslash x_0$, such that $f_0(x) = f_k(x)$ for $x \in B_k \backslash B_{k+1}$ and all k and $f_0(x_1) = x$ for all others x is also bilipschitz and $KI(f_0) < Q(n)$. We can extend bilipschitz homeomorphism f_0 to the point x_0 without change of the dilatation $KI(f_0)$.

The end of the proof is the same as for case 1. □

Theorem 2.3 has some versions. The next result demonstrates one of them.

The homeomorphism $\varphi : U \cong V$ belongs to the class $QC_0(U, V)$ (correspondingly, $LIP_0(U, V)$, $LQC_0(U, V)$), if there is a closed set $A \subset U$ such that $\varphi \restriction (U \backslash A) = \mathrm{Id}$.

Theorem 2.3′. *Let $U \subsetneq \mathbb{R}^n$, $V \subsetneq \mathbb{R}^n$ be open and $\varphi : U \cong V$ be a homeomorphism. Suppose that for each autohomeomorphism $f \in LIP_0(U)$ the autohomeomorphism $f^\varphi = \varphi \circ f \circ \varphi^{-1}$ belongs to $LQC(U)$. Then φ is locally quasiconformal.*

This is a light variation of Theorem 2.3. In the proof of the Theorem 2.3 we really used only the class $LIP_0(U)$.

2.5. *The main theorem (local version)*

Theorem 2.4. *For $m, n > 1$ let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open, and suppose that τ is a group isomorphism between the groups $LQC(U)$ and $LQC(V)$. Then there is $\varphi : U \cong V$ such that φ is LQC and φ induces τ. That is, for every $f \in LQC(U)$:*

$$\tau(f) = f^{\varphi} \quad (f^{\varphi} = \varphi \circ f \circ \varphi^{-1}).$$

Proof. By Theorem 2.2 there is $\varphi : U \cong V$ such that φ includes τ. By Theorem 2.3 the homeomorphism φ is locally quasiconformal. $\square$

3. Global versions of the main theorem

In the paper [3] an example of a locally bilipschitz homeomorphism that induces isomorphism between two groups of bilipschitz autohomeomorphisms was constructed. In the case of groups of quasiconformal autohomeomorphisms this effect does not exist. The isomorphism between two groups of quasiconformal homeomorphism induce quasiconformal homeomorphism. Remember that each quasiconformal homeomorphism $\varphi : U \cong V$ between two domains $U, V \subset \mathbb{R}^n$ induces an isomorphism $\tau : QC(U) \longleftrightarrow QC(V)$. Therefore we obtained necessary and sufficient conditions for isomorphism of groups $QC(U)$ and $QC(V)$.

3.1. *The main theorem for homeomorphisms*

Theorem 3.2. *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$ be open and $\varphi : U \cong V$ be a homeomorphism. Suppose that for each autohomeomorphism $f \in LIP(U)$ the autohomeomorphism $f^{\varphi} = \varphi \circ f \circ \varphi^{-1}$ belong to $QC(V)$. Then φ is quasiconformal.*

Proof. By Theorem 2.3 the homeomorphism φ is locally quasiconformal. We will prove that φ is quasiconformal. Suppose by contradiction that φ is not quasiconformal. Hence there is a sequence of points $\{x_k \subset U\}$ such that $K_{\varphi}(x_k) \longrightarrow \infty$ if $k \longrightarrow \infty$. Choose a sequence of balls $\{B_k = B(x_k, r_k)\}$ with the following properties:

$$B_k \cap B_{k_1} = \emptyset \quad \text{if } k \neq k_1,$$
$$V_k \stackrel{\text{def}}{=} B(\varphi(x_k), 2L_{\varphi}(x_k, r_k)) \subset V.$$

In the interior of each ball B_k we choose a new ball $\hat{B}_k = B(x_k, \rho_k) \subset B(x_k, \frac{1}{2}r_k)$ such that $Q_k = K_{\varphi}(x_k, \rho_k) > \frac{1}{2}K_{\varphi}(x_k)$. So $\lim_{k\to\infty} Q_k = \infty$. Remember that $Q_k = L_k/l_k$ where $L_k = L_{\varphi}(x_k, \rho_k)$, $l_k = l_{\varphi}(x_k, \rho_k)$. For each k choose two points u_k, $z_k \subset \partial\hat{B}_k$ such that

$$|\varphi(u_k) - \varphi(x_k)| = L_k, \quad |\varphi(z_k) - \varphi(x_k)| = l_k.$$

Let $\theta_k \in SO_n(x_k)$ and $\theta_k(z_k) = u_k$. By Lemma 1.3 there is a diffeomorphism $f_k : U \cong U$, $f_k\lceil\hat{B}_k = \theta_k$, $f_k\lceil(U\backslash B_k) = \mathrm{Id}$, $KI(f_k) < Q(n)$, where the number $Q(n)$ depends only on n. Let $f_0 : U \cong U$ be a bilipschitz homeomorphism such that $f_0(x) = f_k(x)$ for $x \in B_k$ and for all k and $f_0(x) = x$ for all other x. It is evident that $KI(f_0) < Q(n)$. By conditions of the theorem, $g_0 = f_0^{\varphi}$ is quasiconformal, i.e. $Q_0 = K(g_0) < \infty$. Remember that $f_0\lceil B_k = f_k$. Hence $g_k = f_k^{\varphi}$ is quasiconformal and $Q_k = K(g_k) \leqslant Q_0$. Let $V_k = \varphi(B_k)$, $S_k = \varphi(\hat{B}_k)$. Then g_k has the following properties:

a) $g_k(V_k) = V_k$;
b) $g_k(S_k) = S_k$;
c) $g_k\lceil(V\backslash\varphi(B_k)) = \mathrm{Id}$;
d) $g_k(\varphi(z_k)) = \varphi(u_k)$;
e) $g_k(\varphi(x_k)) = \varphi(x_k)$.

From Theorem 2.1 it follows, that

$$K_{g_k}(\varphi(x_k),\rho) = \frac{L_{g_k}(\varphi(x_k),\rho)}{l_{g_k}(\varphi(x_k),\rho)} < P < \infty,$$

where a constant P depends only on n and Q_0. The inequality (3.1) is correct for all $\rho \in (0, L_{\varphi}(x_k,\rho_k))$. For sets V_k, S_k, points $\varphi(x_k)$, $\varphi(u_k)$, $\varphi(z_k)$, for the homeomorphism g_k, for numbers $r_0 = l_k$, $r_1 = L_k$, $k_1 = Q_k$, $r_2 = 2L_k$ and for an arbitrary point

$$z^* \subset V_k\backslash\overline{B(\varphi(x_k), L_k)},$$

all conditions of the Lemma 2.4 hold, so

$$K_{g_k}(\varphi(x_k),\rho_k) > Q_k.$$

This conclusion contradicts to (3.1), so φ is quasiconformal. □

3.3. *Proof of the main theorem (Theorem 0.1)*

Theorem 3.4. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open. Groups $QC(U)$ and $QC(V)$ are isomorphic iff there is a quasiconformal homomorphism $\varphi : U \cong V$.*

Proof. If groups $QC(U)$ and $QC(V)$ are isomorphic then by Theorem 2.3 there is $\varphi \in LQC(U,V)$ that induces τ. By Theorem 3.2, φ is QC.

If there is a quasiconformal homeomorphism $\varphi : U \cong V$ then $\tau = f^{\varphi}$ is an isomorphism between $QC(U)$ and $QC(V)$. □

Note. Of course, for the proof of the Theorem 3.2 we used only some special subgroup of $LIP(U)$. This subgroup contains all "quasitranslations" and "quasi-rotations" of two types (see Lemma 1.2, 1.3, 1.4) and its compositions and limits.

4. Remarks about manifolds

4.1. *Local version of the main theorem*

4.1 Definitions. A metric space M is an n-dimensional *Lipschitz manifold* if for each point $x \in M$ there is an open set $U \ni x$ and a bilipschitz homeomorphism $\varphi : B(0,1) \cong U$.

A metric space M is an n-dimensional *quasiconformal manifold* if for each point $x \in M$ there is an open set $U \in x$ and quasiconformal homeomorphism $\varphi : B(0,1) \cong U$.

Local versions of the main theorem are correct for Lipschitz (quasiconformal) manifolds also:

Theorem 4.1. *Let M, N be Lipschitz manifolds and $\varphi : M \cong N$ be a homeomorphism. Suppose that for each homeomorphism $f \in LIP(M)$, the autohomeomorphism $f^{\varphi} = \varphi \circ f \circ \varphi^{-1}$ belongs to $LQC(N)$. Then φ is locally quasiconformal.*

Theorem 4.2. *Let M, N be quasiconformal manifolds and suppose that τ is a group isomorphism between the groups $LQC(M)$ and $LQC(N)$. Then there is $\varphi : M \cong N$ such that φ is LQC and φ induces τ.*

Proofs of Theorem 4.1, 4.2 are the same as for the case of Euclidean space.

4.2. *Global versions of the main theorem*

Definitions. A metric space M is an n-dimensional *L-Lipschitz manifold* if for each point $x \in M$ there is an open set $U \ni x$ and a bilipschitz homeomorphism $\varphi : B(0,1) \cong U$ such that $KI(\varphi) \leqslant L$. The constant L does not depend on x.

A metric space M is an n-dimensional *Q-quasiconformal manifold* if for each point $x \in M$ there is a open set $U \ni x$ and quasiconformal homeomorphism $\varphi : B(0,1) \cong U$ such that metric dilatation $K(\varphi)$ of φ is less then Q. The constant Q does not depend on x.

Theorem 4.3. *Let M be a L-Lipschitz manifold, N be a Q-quasiconformal manifold and $\varphi : M \cong N$ be a homeomorphism. Suppose that for each $f \in LIP(M)$ the autohomeomorphism $f^{\varphi} = \varphi \circ f \circ \varphi^{-1} \in QC(N)$. Then φ is quasiconformal.*

Proof. By Theorem 4.2, φ is LQC. We will prove that it is quasiconformal. Suppose by contradiction that φ is not quasiconformal. Hence there is a sequence of points $\{x_k \in M\}$ such that $K_{\varphi}(x_k) \to \infty$ if $k \to \infty$. By the definition of L-Lipschitz manifold there is a sequence of open sets $\{U_k \ni x_k\}$ and bilipschitz homeomorphisms $\varphi_k : B(0,1) \to U_k$ such that $KI(\varphi_k) \leqslant L$. On the other hand, by the definition of Q-quasiconformal manifold, there is a sequence of open sets $\{V_k \ni \varphi(x_k)\}$ and quasiconformal homeomorphisms $\psi_k : V_k \cong B(0,1)$ such that $K(\psi_k) \leqslant Q$.

Choose a sequence of balls $B_k = B(\varphi_k^{-1}(x_k), r_k)$, $r_k = 1$, with the following properties:

a) $\varphi(\varphi_k(B_k)) \cap \varphi(\varphi_{k_1}(B_{k_1})) = \emptyset$ if $k \neq k_1$

b) $V_k = B((\theta_k \circ \varphi)(x_k)), 2L_{\theta_k \circ \varphi \circ \varphi_k}(\varphi \circ \varphi_k(x_k), r_k) \subset B(0,1)$.

Similarly to the proof of Theorem 3.2, we construct quasi-rotations $\tilde{f}_k : B(0,1) \cong B(0,1)$ such that $\tilde{f}_k \lceil (B(0,1) \backslash B_k) = \mathrm{Id}$. The homeomorphisms $f_k = \varphi_k \circ f_k \circ \varphi_k^{-1}$ are bilipschitz. It is possible to extend f_k as an identity on all manifold N. Composition $f_0 = f_1 \circ f_2 \circ \ldots \circ f_k \circ \ldots$ is only formal, because at each point x where $f_k(x) \neq x$ all other homeomorphisms are identities. So f_0 is a bilipschitz homeomorphism. By supposition of the Theorem, $g_0 = f_0^{\varphi}$ is quasiconformal. Because N is a Q-quasiconformal manifold, all homeomorphisms $\tilde{g}_k = \theta_k \circ g_0$ are also quasiconformal and $K(\tilde{g}_k) \leqslant Q \cdot K(g_0)$.

Similarly to the proof of Theorem 3.2 we can prove that $K_{g_k}(\theta \circ \varphi(x_k)) \to \infty$ for $k \to \infty$. This conclusion contradicts the inequality $K(\tilde{g}_k) < Q \cdot K(g_0)$.

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